Curvature: how quickly does a curve change direction?

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| \text{ is curvature.}$$

$\vec{T} = \vec{T}(s)$ ← arc length } independent of choice of parameterization

$\left|\frac{d\vec{T}}{ds}\right| = ??$ ... apply Chain Rule

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\cdot\frac{ds}{dt} \Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|ds/dt|}$$

} Easier to find.

Recall $\frac{ds}{dt} = |\vec{r}\,'(t)|$

$$\Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

OK, but we still need $\underbrace{\frac{d}{dt}\left(\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right)}_{\text{can be tough}}$

It turs out there is another formula:

$$\boxed{K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}.}$$

EX: Find curvature of $\vec{r}(t)=\langle 1+t, 2t, t^2\rangle$.

You could try finding $\vec{T}'(t)$... it will get ugly fast. Quicker to use the other formula: $\kappa = |\vec{r}\,'\times\vec{r}\,''| / |\vec{r}\,'|^3$.

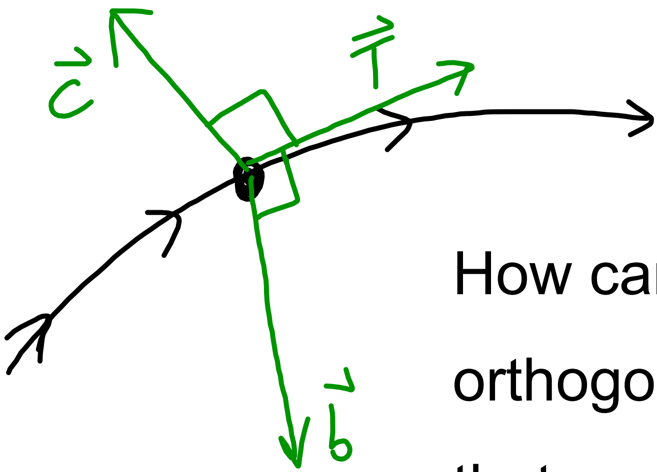
$\vec{r}\,' = \langle 1, 2, 2t\rangle$ ... $|\vec{r}\,'| = \sqrt{5+4t^2}$.

$\vec{r}\,'' = \langle 0, 0, 2\rangle$

$\vec{r}\,'\times\vec{r}\,'' = \langle 4, -2, 0\rangle \Rightarrow |\vec{r}\,'\times\vec{r}\,''| = \sqrt{16+4} = 2\sqrt{5}$

$$\Rightarrow \kappa = \frac{2\sqrt{5}}{(5+4t^2)^{3/2}}.$$

The normal and binormal vectors



How can we find three orthogonal unit vectors that move with a particle through space?

$\vec{T}$ points in the direction of motion, so that will be the first vector. Note that $|\vec{T}|=1 \Rightarrow |\vec{T}|^2=\vec{T}\cdot\vec{T}=1$

$$\Rightarrow \frac{d}{dt}(\vec{T}\cdot\vec{T})=\frac{d}{dt}(1)=0$$

$$\Rightarrow 2\vec{T}\cdot\vec{T}'=0$$

$$\Rightarrow \underline{\vec{T}\perp\vec{T}'}$$

so normalize this...

$$\vec{N}=\frac{\vec{T}'}{|\vec{T}'|}.$$

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$: NORMAL vector
for a curve (not a plane).

Bi-normal : $\vec{B} = \vec{T} \times \vec{N}$

$|\vec{B}| = |\vec{T}||\vec{N}| \sin(90°) = 1 \cdot 1 \cdot 1 = \underline{1}$.

$\{\vec{T}, \vec{N}, \vec{B}\}$: the "TNB frame".

EX: Find the TNB frame for

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ @ $t = \pi/2$.

$$\left.\begin{aligned} \vec{r}\,' &= \langle -\sin(t), \cos(t), 1 \rangle \\ |\vec{r}\,'| &= \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2} \end{aligned}\right\} \vec{T} = \frac{1}{\sqrt{2}} \vec{r}\,'$$

$$\vec{T}\,'(t) = \frac{1}{\sqrt{2}} \langle -\cos(t), -\sin(t), 0 \rangle \Big\} |\vec{T}\,'| = \frac{1}{\sqrt{2}}$$

So $\vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$.

Plug in $t = \pi/2$...

$$\vec{T} = \frac{1}{\sqrt{2}}\left\langle -\sin\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{2}\right), 1\right\rangle = \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle.$$

$$\vec{N} = \left\langle -\cos\left(\frac{\pi}{2}\right), -\sin\left(\frac{\pi}{2}\right), 0\right\rangle = \langle 0, -1, 0\rangle$$

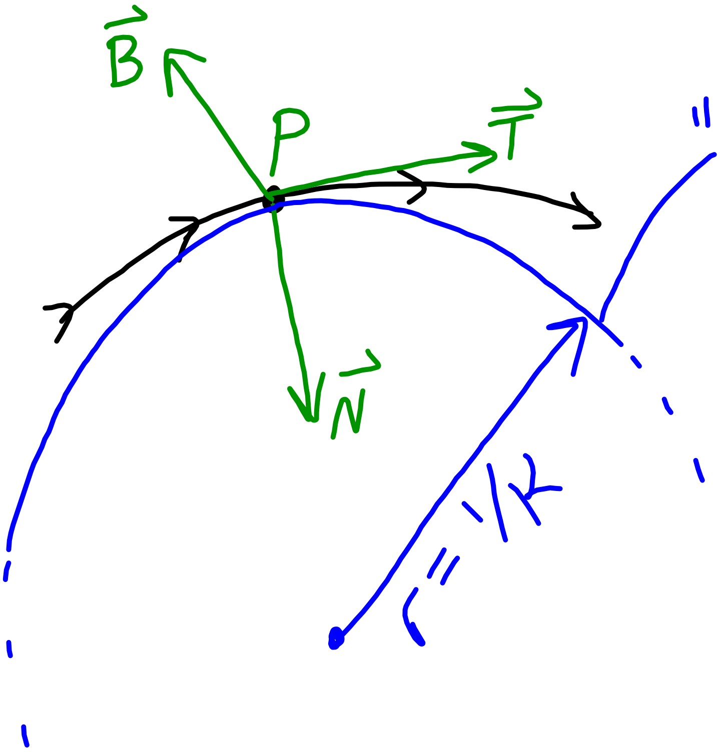
$$\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle.$$

SUMMARY

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} \qquad \vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} \qquad K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}.$$

$$\vec{B} = \vec{T} \times \vec{N}$$

The osculating plane – contains $\vec{T}$ & $\vec{N}$



"osculating circle"

- Tangent to curve at P.
- Lies in the osculating plane.
- Same $\vec{T}, \vec{N}, \vec{B}$ and κ!

Velocity and acceleration

$\vec{r}(t)$: position at time t.

$\vec{v}(t) = \vec{r}\,'(t)$; velocity at time t.

$$\vec{v} = |\vec{v}|\,\hat{v}$$

speed direction

$$\vec{a}(t) = \vec{v}\,'(t)$$
$$= \vec{r}\,''(t)$$; acceleration

Given acceleration, we can also find velocity and position by integrating...

$$\vec{v} = \int \vec{a}(t)\,dt$$

$$\vec{r} = \int \vec{v}(t)\,dt$$

We require unknown constants to be added at each step; their values are often determined later from "initial conditions".

EX: Let $\vec{a} = \langle 100, 100, -9.8 \rangle$ (m/s^2)

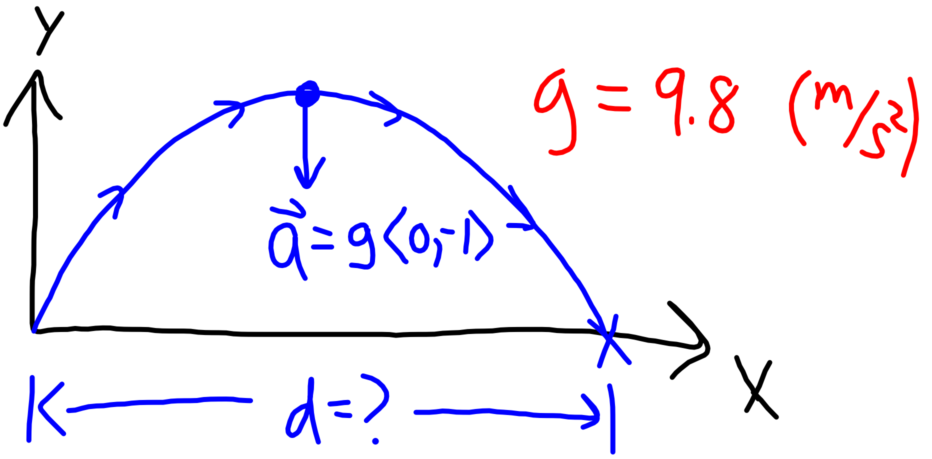
Find position at time t.

$$\vec{v}(t) = \int \vec{a}\,dt = t\langle 100, 100, -9.8 \rangle + \vec{C}_1$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \frac{1}{2}t^2\langle 100, 100, -9.8 \rangle + \vec{c}_1 t + \vec{C}_2.$$

*Note $\vec{r}(0) = \vec{C}_2$ & $\vec{v}(0) = \vec{C}_1$.

EX: A projectile is fired from a cannon at a 45 degree angle across flat terrain. The muzzle velocity is 300 m/s. Neglecting air friction, find the distance away that the projectile lands.



$$\vec{v}(t) = t\langle 0, -9.8\rangle + \vec{v}(0)$$

$$= t\langle 0, -9.8\rangle + \frac{300}{\sqrt{2}}\langle 1, 1\rangle.$$

$$\vec{r}(t) = \int \vec{v}\,dt = \frac{1}{2}t^2\langle 0, -9.8\rangle + 150\sqrt{2}\,t\langle 1, 1\rangle + \cancel{\vec{r}(0)} \rightarrow \vec{0}$$

Look for y-component of $\vec{r}(t)$ to be zero (height of projectile):

$$-4.9t^2 + 150\sqrt{2}\,t = 0$$

$$t\left(150\sqrt{2}-4.9t\right)=0$$

$$t=\frac{150\sqrt{2}}{4.9}.\quad (\text{seconds})$$

$$\text{Distance} = x\left(\frac{150\sqrt{2}}{4.9}\right)$$

$$=\frac{\left(150\sqrt{2}\right)^2}{4.9}\quad (\text{meters})$$

$$\approx 9.18\text{ Km}$$

Newton's Second Law

$$\vec{F} = m\vec{a}$$

force mass

Often we just want the acceleration:

$$\vec{a} = \frac{1}{m}\vec{F}.$$

EX : Consider a particle of mass m=11,110 kg orbiting the origin according to

$$\vec{r}(t) = |\vec{r}| \langle \cos(\omega t), \sin(\omega t) \rangle$$

$$|\vec{r}| = 6.9113 \cdot 10^6 \text{ m}$$

$$\omega = \frac{2\pi}{5790} s^{-1}$$

Find the force acting on the particle.

(represents the Hubble Space Telescope)

$\vec{F} = m\vec{a}$... need $\vec{a}$.

$\vec{r} = |\vec{r}|\langle \cos(\omega t), \sin(\omega t)\rangle$

$\vec{v} = \vec{r}\,' = |\vec{r}|\omega\langle -\sin(\omega t), \cos(\omega t)\rangle.$

$\vec{a} = \vec{v}\,' = -|\vec{r}|\omega^2 \underbrace{\langle \cos(\omega t), \sin(\omega t)\rangle}_{\text{unit vector}}.$

$\vec{F} = m\vec{a} \Rightarrow |\vec{F}| = |\vec{r}|m\omega^2$

$\approx 90.4\,\text{kN}.$

Practice!

#1 Find K for $\vec{r} = \langle t, 2t, e^t \rangle$.

$\vec{r}\,' = \langle 1, 2, e^t \rangle$ $\quad |r'| = \sqrt{5+e^{2t}}$

$\vec{r}\,'' = \langle 0, 0, e^t \rangle$

$|\vec{r}\,' \times \vec{r}\,''| = |\langle 2e^t, -e^t, 0 \rangle| = \sqrt{4e^{2t}+e^{2t}}$

$$K = \frac{e^t\sqrt{5}}{(5+e^{2t})^{3/2}}.$$

#2 Find the equation of the osculating plane through $(-\sqrt{3}, 0, 2\pi+4)$ for the curve $\vec{r}(t) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t), 2t+4 \rangle$.

$$\vec{r}\,' = \langle -\sqrt{3}\sin(t), \sqrt{3}\cos(t), 2 \rangle$$

$$|\vec{r}\,'| = \sqrt{3(\sin^2(t)+\cos^2(t))+4} = \sqrt{7}$$

$$\vec{T} = \frac{1}{\sqrt{7}}\langle -\sqrt{3}\sin(t), \sqrt{3}\cos(t), 2 \rangle.$$

$$\vec{T}\,' = \frac{1}{\sqrt{7}}\langle -\sqrt{3}\cos(t), -\sqrt{3}\sin(t), 0 \rangle.$$

$$\vec{N} = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\Rightarrow \vec{T} = \langle 0, -\sqrt{3}/7, \frac{2}{\sqrt{7}} \rangle$$

$$\vec{N} = \langle 1, 0, 0 \rangle$$

$$\vec{B} = \langle 0, \frac{2}{\sqrt{7}}, \sqrt{3}/7 \rangle$$

$$\vec{B} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

$$\Rightarrow \frac{2y}{\sqrt{7}} + \sqrt{\frac{3}{7}}\left(z - (2\pi + 4)\right) = 0$$

#3 Let a particle of mass 10 kg be acted upon by force

$$\vec{F} = \langle -10, 10, 20 \rangle \text{ (newtons).}$$

If initial velocity is $\langle 1, 0, 0 \rangle$ and it starts at the origin, find $\vec{r}(t)$.

$$\vec{a} = \frac{1}{m}\vec{F} = \langle -1, 1, 2 \rangle$$

$$\vec{v} = \langle -1, 1, 2 \rangle t + \langle 1, 0, 0 \rangle$$

$$\vec{r} = \frac{1}{2}t^2 \langle -1, 1, 2 \rangle + t \langle 1, 0, 0 \rangle + \cancelto{\vec{0}}{\vec{r}(0)}$$